Mathematics - Course 321

APPENDIX 3: SOLVING QUADRATIC EQUATIONS

I Introduction

A *quadratic function* is a function of the form

$$f(x) = ax^2 + bx + c,$$

where a, b, c are real constants.

A *quadratic equation* is an equation of the form

$$ax^2 + bx + c = 0$$

II Roots of a Quadratic Equation

The *roots of a quadratic equation* are the x-values which *satisfy* the equation. Therefore, to *solve* a quadratic equation is to find its roots.

The roots of the quadratic equation,

$$ax^2 + bx + c = 0$$

are given by the formula,

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

The quantity b^2-4ac is called the *discriminant*, designated "D". The value of D determines the number and nature of roots of the quadratic, as summarized in the following table:

Value of $D = b^2-4ac$	Number and Nature of Roots	Figure
>0	2 real roots	1,2
0	1 real root	3,4
<0	2 complex roots	5,6

III Graphical Solution of Quadratic Equations

The graph of a quadratic function $y = ax^2 + bx + c$ is a *parabola*, and the roots of the corresponding quadratic equation, $ax^2 + bx + c = 0$, are the x-coordinates of the parabola's x-intercepts, since $y = 0$ at these x-values.

The "ax^2" term dominates the value of $ax^2 + bx + c$ for large values of x, and therefore, the sign of "a" governs whether the parabola opens upward or downward, as summarized in the following table:

Value of "a"	Parabola Opens	As in Figures
> 0	upward	1, 3, 5
< 0	downward	2, 4, 6

The graphical significance of "a" and "D" values is summarized in Figures 1 to 6.

IV Examples

(1) $y = 2x^2 - 3x - 4$
(2) $y = -x^2 + 6x - 9$
(3) $y = x^2 + x + 2$

(a) Graph each of the above quadratic functions.

(b) From the graphs, find the roots of the corresponding quadratic equations.

(c) Calculate the roots by formula.

Summary of Graphical Significance of Values of "a" and "D" = $b^2 - 4ac$"

a > 0		a < 0
Figure 1	2 real roots x_1, x_2 (D > 0)	**Figure 2**

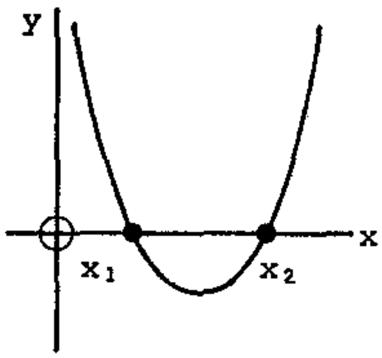
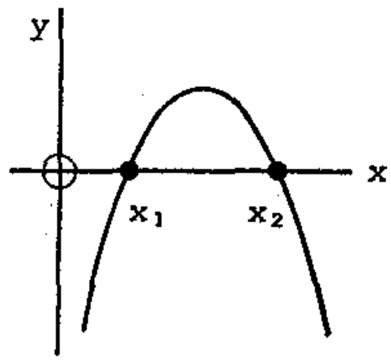

		

Figure 3	1 real root, x_1 (D = 0)	**Figure 4**

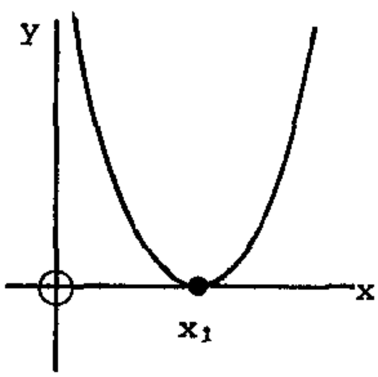
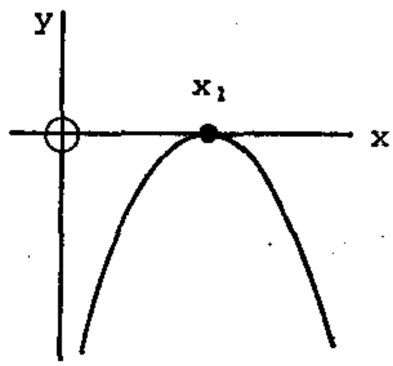
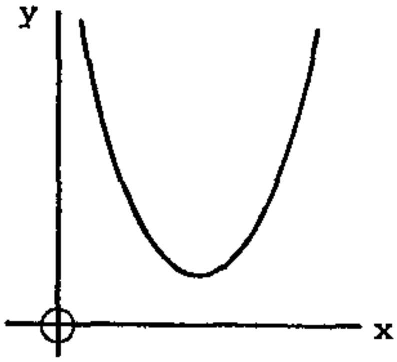
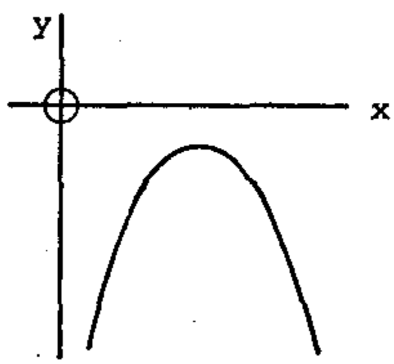
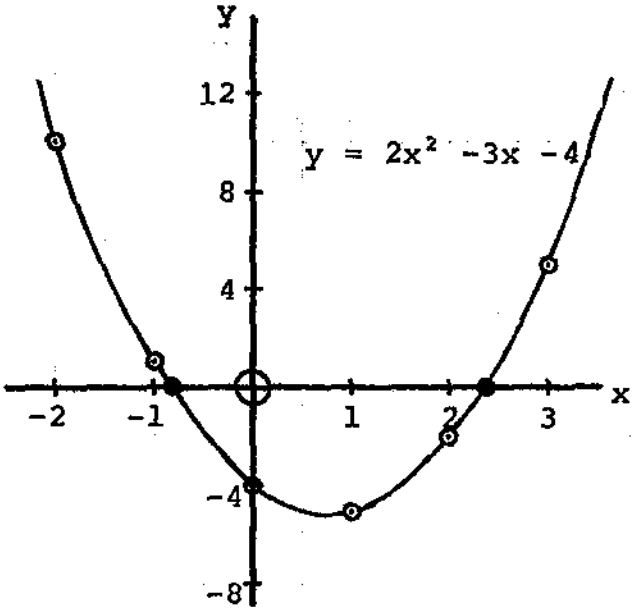

		

Figure 5	2 complex roots (D < 0)	**Figure 6**
		

Solutions to Examples:

1. (a)

x	-2	-1	0	1	2	3
y	10	1	-4	-5	-2	5



(b) From graph, roots of $2x^2 - 3x - 4$ are approximately -0.8, 2.3.

(c) $a = 2,\ b = -3,\ c = -4$

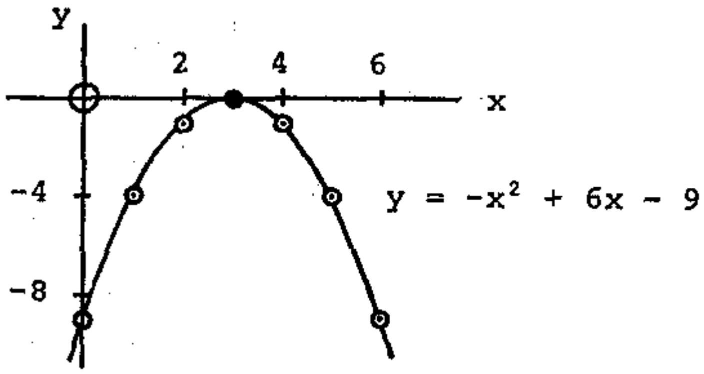
$$\therefore \text{ roots are } x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-4)}}{2(2)}$$

$$= \frac{3 \pm \sqrt{41}}{4}$$

$$= 2.35 \text{ or } -0.85 \quad \text{(to 2 D.P.)}$$

2. (a)

x	0	1	2	3	4	5	6
y	-9	-4	-1	0	-1	-4	-9



(b) The graph indicates that $-x^2 + 6x - 9 = 0$ has only one root, namely 3.

(c) $a = -1$, $b = 6$, $c = -9$

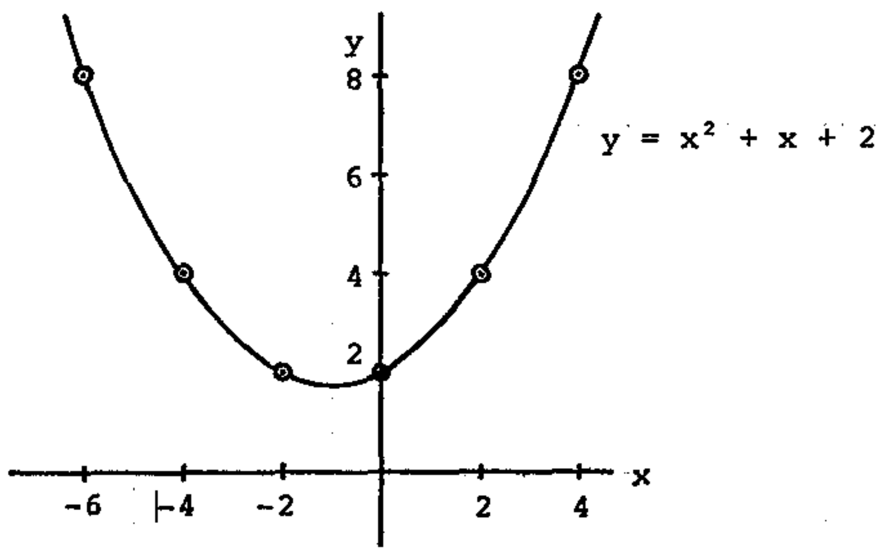
$$\therefore \text{ roots are } x = \frac{-6 \pm \sqrt{6^2 - 4(-1)(-9)}}{2(-1)}$$

$$= \frac{-6 \pm \sqrt{0}}{-2}$$

$$= 3$$

3. (a)

x	-3	-2	-1	0	1	2
y	8	4	2	2	4	8



(b) Since the parabola does not intersect the x-axis, $x^2 + x + 2 = 0$ has no real roots.

(c) $a = 1, b = 1, c = 2$

$\therefore$ roots are $$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(2)}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{-7}}{2}$$

Since $b^2 - 4ac < 0$, roots are complex.

V Other Methods of Solving Quadratics

Alternative methods for solving quadratic equations include

(1) factoring, and
(2) completing the square.

Trainees are not required to be able to use these alternative methods, but may use such methods at their discretion.

ASSIGNMENT

Solve each of the following quadratic equations

(a) by graphing the associated quadratic function

(b) by using the formula.

1. $x^2 - 3 = 0$
2. $x^2 - 2x - 8 = 0$
3. $4x^2 - 15x + 9 = 0$
4. $9x^2 - 24x + 16 = 0$
5. $-x^2 + 5x + 2 = 0$
6. $2x^2 + x + 3 = 0$

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